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1990 J. Phys. A: Math. Gen. 23 751

(http://iopscience.iop.org/0305-4470/23/5/018)

J. Phys. A: Math. Gen. **23** (1990) 751-760. Printed in the UK

Hall–Littlewood symmetric functions and the BKP equation

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Received 18 July 1989

Abstract. The connection between solutions of the BKP equation and Hall-Littlewood symmetric functions is utilised in a unified approach to soliton and polynomial solutions. This is analogous to the Wronskian formulation of the solution of the KP equation. As a by-product, two novel expressions for certain Hall-Littlewood functions in terms of Pfaffians are derived.

1. Introduction

It is well known that both polynomial and soliton-type solutions to equations associated with the bilinear KP hierarchy (Sato 1981) can be conveniently expressed in terms of Wronskian determinants (Freeman and Nimmo 1983, Ohta *et al* 1988, Nimmo 1989). A second hierarchy, the BKP hierarchy, was discovered by Date *et al* (1982) and its polynomial solutions were determined. These solutions are expressed as Pfaffians rather than determinants. Recently, You (1989) has given a more explicit description of these solutions in terms of Schur's Q functions, which arise in the theory of projective representations of the symmetric groups (Schur 1911). Also, very recently, Hirota (1989) has shown how solutions to the BKP equation may be verified by direct substitution using a quadratic Pfaffian identity. This work is viewed by Hirota as analogous to the work of Nakamura (1989) on the KP equation, in which solutions are expressed as non-Wronskian determinants. The aim of this paper is to describe a similar method which is directly analogous to the Wronskian approach, and also demonstrates the connection with the work of You (1989), and thereby with Date *et al* (1982).

2. Hall–Littlewood functions

The KP equation in bilinear form is written as

$$(D_{(1^4)}+3D_{(2^2)}-4D_{(31)})\tau\circ\tau=0 \tag{2.1}$$

where τ is a function of a sequence of independent variables $x=x_1,x_2,\ldots$ and D_λ, for any partition $\lambda=(\lambda_1,\ldots,\lambda_n)$, denotes the Hirota derivative

$$D_{x_{\lambda_1}}\ldots D_{x_{\lambda_n}}\tau\circ\tau=\left(\frac{\partial}{\partial x_{\lambda_1}}-\frac{\partial}{\partial x'_{\lambda_1}}\right)\ldots\left(\frac{\partial}{\partial x_{\lambda_n}}-\frac{\partial}{\partial x'_{\lambda_n}}\right)\tau(x)\tau(x')|_{x'=x}.$$

If $\varphi_1,\ldots,\varphi_N$ are any functions of x satisfying

$$\partial_j\varphi_i=\partial_1^j\varphi_i \tag{2.2}$$

0305-4470/90/050751+10$03.50

for $i=1,\ldots,N$ and $j\in\mathbb{Z}^+$, where $\partial_k=\partial/\partial x_k$, then the Wronskian determinant $\tau=W(\varphi_1,\ldots,\varphi_N)$ satisfies (2.1) (Freeman and Nimmo 1983). In particular, we may obtain functions $h_j(\boldsymbol{x})$ satisfying (2.2) from the generating function.

$$\prod_{i=1}^{n}(1-\alpha_i z)^{-1}=\exp\left(\sum_{j=1}^{\infty}x_j z^j\right)=\sum_{k=0}^{\infty}h_k(\boldsymbol{x})z^k \tag{2.3}$$

where $\alpha_1,\ldots,\alpha_n$ are fictitious indeterminates such that $x_i=p_i(\boldsymbol{x})/i=(\alpha_1^i+\ldots+\alpha_n^i)/i$ and the polynomials $h_k(\boldsymbol{x})$ may be regarded as the kth complete symmetric function and $p_i(\boldsymbol{x})$ the ith power-sum symmetric function in $\alpha_1,\ldots,\alpha_n$, or as polynomials in $\boldsymbol{x}$. The solutions to (2.1) obtained in this way are Schur functions, explicitly

$$S_\lambda(\boldsymbol{x})=\det(h_{\lambda_i-i+j}(\boldsymbol{x})) \tag{2.4}$$

for any partition $\lambda=(\lambda_1,\ldots,\lambda_n)$.

The generating function (2.3) may be generalised by the introduction of a parameter $t\neq 1$ to give

$$\prod_{i=1}^{n}\frac{(1-t\alpha_i z)}{(1-\alpha_i z)}=\exp\left(\sum_{j=1}^{\infty}(1-t^j)x_j z^j\right)=\sum_{k=0}^{\infty}q_k(\boldsymbol{x};t)z^k \tag{2.5}$$

which reduces to (2.3) when $t=0$. From (2.5) we see that

$$\partial_j q_i=(1-t^j)q_{i-j}=\frac{(1-t^j)}{(1-t)^j}\partial_1^j q_i \tag{2.6}$$

and that if t is a jth root of unity then the q_i are independent of x_k for $k\equiv 0 \bmod j$. Here we use the convention that $q_k=0$ if $k<0$, and observe that $q_0=1$.

The generalisations of the Schur function which arise from the above are Hall-Littlewood symmetric functions Q_λ (Macdonald 1979), which we define in an equivalent way to that of Littlewood (1961) in terms of 'raising operators'. For a partition $\lambda=(\lambda_1,\ldots,\lambda_m)$

$$Q_\lambda(\boldsymbol{x};t)=\prod_{1\leqslant i<j\leqslant m}(\partial^{(i)}-\partial^{(j)})(\partial^{(i)}-t\partial^{(j)})^{-1}\prod_{i=1}^{m}q_{\lambda_i}(\boldsymbol{x}^{(i)};t)|_{\boldsymbol{x}^{(i)}=\boldsymbol{x}} \tag{2.7}$$

which reduces to (2.4) when $t=0$, so that $Q_\lambda(\boldsymbol{x};0)=S_\lambda(\boldsymbol{x})$. In (2.7) $\boldsymbol{x}^{(i)}$ denote distinct copies of $\boldsymbol{x}$ with $\partial^{(i)}=\partial/\partial x_1^{(i)}$. We have

$$(\partial^{(i)}-\partial^{(j)})(\partial^{(i)}-t\partial^{(j)})^{-1}=1+(t-1)\partial^{(i)}\partial^{(j)-2}+(t^2-t)\partial^{(i)2}\partial^{(j)-2}+\ldots \tag{2.8}$$

and we impose boundary conditions such that for $m\geqslant 0$

$$\partial^{-m}q_i=(1-t)^{-m}q_{i+m}$$

so that in particular

$$Q_{(\lambda_1,\lambda_2)}(\boldsymbol{x};t)=q_{\lambda_1}q_{\lambda_2}+(t-1)q_{\lambda_1+1}q_{\lambda_2-1}+\ldots+(t^{\lambda_2}-t^{\lambda_2-1})q_{\lambda_1+\lambda_2}. \tag{2.9}$$

We shall see shortly that polynomial solutions to the BKP equation are given in terms of $Q_\lambda(\boldsymbol{x};1)$ (see also You 1989), but we shall develop the theory as far as possible with arbitrary t.

A related set of polynomials is

$$P_\lambda(\boldsymbol{x};t)=b_\lambda^{-1}(t)Q_\lambda(\boldsymbol{x};t) \tag{2.10}$$

where $b_\lambda(t)=\Pi_{i=1}^{n}\varphi_{m_i}(t)$ and $\varphi_k(t)=(1-t)(1-t^2)\ldots(1-t^k)$ for a partition $\lambda=(n^{m_n}\ldots 1^{m_1})$. In fact, the P_λ and Q_λ form $\mathbb{Z}[t]$ bases for polynomials in x with $\mathbb{Z}[t]$ coefficients and an inner product $\langle\ ,\ \rangle$ is defined, under which $\{P_\lambda\}$ and $\{Q_\lambda\}$ are dual

$$\langle Q_\lambda(\boldsymbol{x};t), P_\mu(\boldsymbol{x};t)\rangle=\delta_{\lambda\mu}. \tag{2.11}$$

Extending the notation for power sum in the usual way for any partitions $\lambda=(\lambda_1,\ldots,\lambda_m)=(n^{m_n},\ldots,1^{m_1})$ and μ, we find that for this inner product

$$\langle p_\lambda(\boldsymbol{x}), p_\mu(\boldsymbol{x})\rangle=z_\lambda(t)\delta_{\lambda\mu} \tag{2.12}$$

where $z_\lambda(t)=z_\lambda\,\Pi_{i=1}^{m}(1-t^{\lambda_i})^{-1}$ and $z_\lambda=\Pi_{i=1}^{n}i^{m_i}m_i!$.

Skew Hall-Littlewood symmetric functions $Q_{\lambda/\mu}$ are defined through this inner product also; for all ν

$$\langle Q_{\lambda/\mu}(\boldsymbol{x};t), P_\nu(\boldsymbol{x};t)\rangle=\langle Q_\lambda(\boldsymbol{x};t), P_\mu(\boldsymbol{x};t)P_\nu(\boldsymbol{x};t)\rangle. \tag{2.13}$$

Using a straightforward generalisation of the case for $t=0$, in the language of Foulkes (1949) the operator $A(P_\mu)$, the adjoint of multiplication by P_μ, is such that $A(P_\mu)Q_\lambda=eeQ_{\lambda/\mu}$ and the adjoint of multiplication by p_ρ is a differential operator. Indeed, from (2.12) one may show that for $\rho=(\rho_1,\ldots,\rho_p)$

$$A(p_\rho(\boldsymbol{x}))=\prod_{i=1}^{p}(1-t^{\rho_i})^{-1}\partial_{\rho_i}. \tag{2.14}$$

Finally, the linear relationships

$$p_\rho(\boldsymbol{x})=\sum_\mu X_\rho^\mu(t)P_\mu(\boldsymbol{x};t) \tag{2.15a}$$

$$Q_\mu(\boldsymbol{x})=\sum_\rho z_\rho^{-1}(t)X_\rho^\mu(t)p_\rho(\boldsymbol{x}) \tag{2.15b}$$

define $X_\rho^\mu(t)$, and when $t=0$ they reduce to the familiar relationships between power sums and Schur functions with $X_\lambda^\mu(0)=\chi_\lambda^\mu$, the characters of the symmetric groups.

Equations (2.15) also hold for the adjoints, and so from (2.14) we have

$$\partial_\rho=\prod_{i=1}^{p}(1-t^{\rho_i})\sum_\mu X_\lambda^\mu(t)A(P_\mu) \tag{2.16}$$

and thus

$$\partial_\rho Q_\lambda(\boldsymbol{x};t)=\prod_{i=1}^{p}(l-t^{\rho_i})\sum_\mu X_\rho^\mu(t)Q_{\lambda/\mu}(\boldsymbol{x};t). \tag{2.17}$$

We may use (2.17) to obtain a formula for the skew Hall-Littlewood functions. Introduce the notation

$$\Pi(\partial^{(i)};t)=\prod_{i<j}(\partial^{(i)}-\partial^{(j)})(\partial^{(i)}-t\partial^{(j)})^{-1} \tag{2.18}$$

so that (2.7) is

$$Q_\lambda(\underset{\sim}{x};t)=\Pi(\partial^{(i)};t)\prod_{i=1}^{n}q_{\lambda_i}(\boldsymbol{x}^{(i)};t)|_{\boldsymbol{x}^{(i)}=\boldsymbol{x}} \tag{2.19}$$

and so

$$\begin{aligned}\partial_\rho Q_\lambda(\boldsymbol{x};t)&=\Pi(\partial^{(i)};t)\left(\prod_{j=1}^{p}\sum_{i=1}^{n}\partial_{\rho_j}^{(i)}\right)\prod_{i=1}^{n}q_{\lambda_i}(\boldsymbol{x}^{(i)};t)|_{\boldsymbol{x}^{(i)}=\boldsymbol{x}}\\&=\left[\prod_{j=1}^{p}\frac{(1-t^{\rho_j})}{(1-t)^{\rho_j}}\right]\left(\Pi(\partial^{(i)};t)p_\rho(\partial^{(i)})\prod_{i=1}^{n}q_{\lambda_i}(\boldsymbol{x}^{(i)};t)\right)\Big|_{\boldsymbol{x}^{(i)}=\boldsymbol{x}}.\end{aligned} \tag{2.20}$$

Now, using (2.15a) and comparing (2.20) and (2.17), we have

$$Q_{\lambda/\mu}(\boldsymbol{x};t)=(1-t)^{-|\mu|}\left(\Pi(\partial^{(i)};t)P_\mu(\partial^{(i)};t)\prod_{i=1}^{n}q_{\lambda_i}(\boldsymbol{x}^{(i)};t)\right)\Big|_{\boldsymbol{x}^{(i)}=\boldsymbol{x}} \tag{2.21}$$

where $|\mu|$ denotes the sum of the parts of μ.

When $t=0$ this becomes the Jacobi-Trudi identity for skew Schur functions (MacDonald 1979) and for $t=-1$ we have an explicit version of (2.21) involving Pfaffians. In fact, we prove in the appendix that

$$\begin{aligned}Q_{\lambda/\mu}(\boldsymbol{x};-1)&=\mathrm{Pf}(Q_{(\lambda_i,-\tilde{\mu}_j,\lambda_j)})\\&=(1\ 2\ldots n\ a_m\ldots a_1)\end{aligned} \tag{2.22}$$

where the notation is also fully explained in the appendix.

3. The BKP equation

In this section we show how some of the results of the previous section may be used to obtain solutions of the BKP equation, in Hirota form

$$(D_{(1^6)}-5D_{(31^3)}-5D_{(3^2)}+9D_{(51)})\tau\circ\tau=0. \tag{3.1}$$

It has been shown (Date *et al* 1982, You 1989) that polynomial solutions of (3.1) are given by

$$\tau=Q_\lambda(2\boldsymbol{x};-1) \tag{3.2}$$

for any partition of λ into distinct parts. We may verify this directly using the Pfaffian identity

$$\begin{aligned}&(1\ 2\ldots n\ n+1\ n+2\ n+3\ n+4)(1\ 2\ldots n)\\&\quad-(1\ 2\ldots n\ n+1\ n+2)(1\ 2\ldots n\ n+3\ n+4)\\&\quad+(1\ 2\ldots n\ n+1\ n+3)(1\ 2\ldots n\ n+2\ n+4)\\&\quad-(1\ 2\ldots n\ n+1\ n+4)(1\ 2\ldots n\ n+2\ n+3)=0\end{aligned} \tag{3.3}$$

(Matsuno 1989, Hirota 1989), where n is even.

Expressing the derivatives of τ given by (3.2) using (2.17) gives

$$\partial_\rho Q_\lambda(2\boldsymbol{x};-1)=\sum_m X^\mu_\rho(-1)Q_{\lambda/\mu}(2\boldsymbol{x};-1) \tag{3.4}$$

and making use of the tables for $X^\mu_\rho(-1)$ which appear in the appendix, we get

$$\begin{aligned}&(D_{(1^6)}-5D_{(31^3)}-5D_{(3^2)}+9D_{(51)})Q_\lambda\circ Q_\lambda\\&\quad=2(\partial_{(1^6)}-5\partial_{(31^3)}-5\partial_{(3^2)}+9\partial_{(51)})Q_\lambda Q_\lambda\\&\qquad-3(2\partial_{(1^5)}-5\partial_{(31^2)}+3\partial_{(5)})Q_\lambda\partial_{(1)}Q_\lambda\\&\qquad+15(\partial_{(1^4)}-\partial_{(31)})Q_\lambda\partial_{(1^2)}Q_\lambda-5(2\partial_{(1^3)}+\partial_{(3)})Q_\lambda(\partial_{(1^3)}-\partial_{(3)})Q_\lambda\\&\quad=90[Q_{\lambda/(321)}Q_\lambda-Q_{\lambda/(32)}Q_{\lambda/(1)}+Q_{\lambda/(31)}Q_{\lambda/(2)}-Q_{\lambda/(21)}Q_{\lambda/(3)}]\\&\quad=90[(1\ldots n\ a_0a_1a_2a_3)(1\ \ldots\ n)-(1\ \ldots\ n\ a_2a_3)(1\ \ldots\ n\ a_0a_1)\\&\qquad+(1\ \ldots\ n\ a_1a_3)(1\ \ldots\ n\ a_0a_2)-(1\ \ldots\ n\ a_1a_2)(1\ \ldots\ n\ a_0a_3)]\end{aligned}$$

which vanishes by virtue of (3.3).

The above verification is, however, not restricted to the polynomials $Q_\lambda(2x;-1)$. Let $\varphi_1(x_0),\ldots,\varphi_n(x_0)$ (n even) be a set of functions depending on $x_0=(x_1,x_3,x_5\ldots)$ and satisfying

$$\partial_k\varphi_i(x_0)=\partial_1^k\varphi_i(x_0) \tag{3.5}$$

for $i=1,\ldots,n$ and k odd. Define the Pfaffian

$$\mathrm{Pf}(\varphi_1,\ldots\varphi_n)=(1\ 2\ \ldots\ n) \tag{3.6}$$

where

$$(ij)=(\partial^{(i)}-\partial^{(j)})(\partial^{(i)}+\partial^{(j)})^{-1}\varphi_i(x_0^{(i)})\varphi_j(x_0^{(j)})|_{x_0^{(i)}=x_0^{(j)}=x_0}.$$

As a consequence of (2.7), we may write (3.6) as

$$\mathrm{Pf}(\varphi_1,\ldots\varphi_n)=\prod_{i<j}(\partial^{(i)}-\partial^{(j)})(\partial^{(i)}+\partial^{(j)})^{-1}\prod_{k=1}^{n}\varphi_k(x_0^{(k)})|_{x_0^{(k)}=x_0}. \tag{3.7}$$

The discussion of section 2 and the proof in the appendix remain valid in this more general case, and we have

$$\partial_\rho\mathrm{Pf}(\varphi_1,\ldots,\varphi_n)=\sum_\lambda X_\rho^\lambda(-1)\ \mathrm{Pf}_\lambda(\varphi_1,\ldots,\varphi_n) \tag{3.8}$$

where $\mathrm{Pf}_\lambda(\varphi_1,\ldots,\varphi_n)=(1\ 2\ldots n\ a_{\lambda_m}\ \ldots\ a_{\lambda_1})$ with (ij) as above, $(ia_{\lambda_j})=\partial^{\lambda_j}\varphi_i$ and $(a_{\lambda_i}a_{\lambda_j})=0$. This is a generalisation of the result of Hirota (1989) with an interpretation for the constants that appear in his work as $X_\mu^\lambda(-1)$.

To obtain soliton solutions, we choose $\varphi_i=\exp[\xi(x_0,p_i)]+c_i\exp[\xi(x_0,q_i)]$ where $\xi(x_0,a)=\sum_{k\,\mathrm{odd}}a^kx_k$ and p_i, q_i and c_i are constants for $i=1,\ldots,n$. The m-soliton solution for m odd is obtained from the $(m+1)$-soliton solution with $p_{m+1}=q_{m+1}=c_{m+1}=0$ so that $\varphi_{m+1}=1$. Notice that

$$(\partial^{(i)}-\partial^{(j)})(\partial^{(i)}+\partial^{(j)})^{-1}\exp[\xi(x_0^{(i)},a)]\exp[\xi(x_0^{(j)},b)]|_{x_0^{(i)}=x_0^{(j)}=x_0}$$
$$=\left(\frac{a-b}{a+b}\right)\exp[\xi(x_0^{(i)},a)]\exp[\xi(x_0^{(j)},b)]$$

and so by (3.7) the one- and two-soliton solutions are

$$P(\varphi_1)=(\partial^{(1)}-\partial^{(2)})(\partial^{(1)}+\partial^{(2)})^{-1}\varphi_1(x_0^{(1)})1|_{x_0^{(i)}=x_0}=\varphi_1(x_0)$$

and

$$P(\varphi_1,\varphi_2)=\frac{p_1-p_2}{p_1+p_2}\exp[\xi(x_0,p_1)+\xi(x_0,p_2)]$$
$$+c_2\frac{p_1-q_2}{p_1+q_2}\exp[\xi(x_0,p_1)+\xi(x_0,q_2)]$$
$$+c_1\frac{q_1-p_2}{q_1+p_2}\exp[\xi(x_0,q_1)+\xi(x_0,p_2)]$$
$$+c_1c_2\frac{q_1-q_2}{q_1+q_2}\exp[\xi(x_0,q_1)+\xi(x_0,q_2)]$$

respectively.

These expressions are the same as the more usual ones (Date *et al* 1982) apart from an unimportant exponential factor. For example, with appropriate choices of c_1 and c_2,

$$P(\varphi_1, \varphi_2) \propto 1 + c_1' \exp[\xi(x_0, p_1) - \xi(x_0, q_1)] + c_2' \exp[\xi(x_0, p_2) - \xi(x_0, q_2)]$$
$$+ c_1' c_2' \frac{(p_1 - p_2)(q_1 - q_2)(p_1 + q_2)(q_1 + p_2)}{(p_1 + p_2)(q_1 + q_2)(p_1 - q_2)(q_1 - p_2)}$$
$$\times \exp[\xi(x_0, p_1) + \xi(x_0, p_2) - \xi(x_0, q_1) - \xi(x_0, q_2)].$$

The representation of the soliton solutions as Pfaffians that we give is different from that of Hirota (1989) and the form (3.7) is analogous to the Wronskian representation of solutions to the KP equation

$$\tau = W(\varphi_1, \dots \varphi_n) = \prod_{i<j} (\partial^{(i)} - \partial^{(j)}) \prod_{k=1}^{n} \varphi_k(x^{(k)})|_{x^{(k)}=x}.$$

Acknowledgment

I would like to thank Chris Athorne for fruitful discussions.

Note added in proof. Since this paper was submitted, the preprint 'A determinantal formula for skew Q-functions' by T Jozefiak and P Pragacz has been brought to the authors' attention. This work also contains a proof of the formula (2.22).

Appendix

Here we record some properties of Pfaffians and establish (2.22).

Let A be an $n \times n$ skew-symmetric matrix with entries a_{ij}. It is known that if n is odd then $\det(A)$ is zero, and if $n = 2m$ is even then $\det(A)$ is a perfect square. In fact, if we define the Pfaffian of A

$$\mathrm{Pf}(A) = \sum_{w \in \tilde{S}_{2m}} \varepsilon(w) w(a_{1,2} a_{3,4} \dots a_{2m-1} a_{2m}) \tag{A1}$$

where $\tilde{S}_{2m}$ denotes the set of permutations of indices $\{1, \dots 2m\}$ such that

$$w(1) < w(2), \dots, w(2m-1) < w(2m) \qquad \text{and} \qquad w(1) < w(3) < \dots < w(2m-1) \tag{A2}$$

and $\varepsilon(w)$ its parity, then

$$\det(A) = (\mathrm{Pf}(A))^2.$$

A frequently used notation for Pfaffians (Caianiello 1973) is

$$\mathrm{Pf}(A) = (1\ 2\ \dots\ 2m) = \sum_{w \in \tilde{S}_{2m}} \varepsilon(w) w((1\ 2)(3\ 4)\ \dots\ (2m-1\ 2m)) \tag{A3}$$

where $(ij) = a_{i,j}$. In the same notation

$$\det(A) = \begin{pmatrix} 1 & 2 \dots 2m \\ 1 & 2 \dots 2m \end{pmatrix} \tag{A4}$$

where the top line indicates the row indices and the second the columns indices. An extension of the above notation is used to denote Pfaffians of matrices with blocks of zeros; one writes

$$\mathrm{Pf}(A)=(\dot{1}\ \dot{2}\ldots\dot{p}\ p+1\ldots 2m) \tag{A5}$$

if the matrix A is such that $a_{ij}=0$ for $1\leqslant i,j\leqslant p$.

Pfaffians have similar symmetry properties and expansion rules to determinants. For any permutation w of indices $\{i_1,\ldots,i_k\}$

$$(w(i_1),w(i_2),\ldots,w(i_k))=\varepsilon(w)(i_1\ldots i_k) \tag{A6}$$

$$(1\ 2\ldots 2m)=\sum_{i=1}^{2m}(-1)^{i+j+1}(ij)(1\ldots i-1,i+1,\ldots j-1,j+1,\ldots 2m) \tag{A7}$$

is the expansion of Pf(A) by its jth line—those entries in A having either index equal to j. The rule for expanding a Pfaffian by a group of k lines is rather more complicated than its determinantal analogue (Caianiello 1973), so we only write it down for the special case we will be interested in.

For a matrix A with $k\times k$ block of zeros Pf(A) is given by

$$(\dot{1}\ldots\dot{k}\ k+1\ldots 2m)$$
$$=(-1)^{k(k-1)/2}\sum_{w\in\hat{S}_{2m-k,k}}\varepsilon(w)\begin{pmatrix}1\ \ldots\ k\\ w(k+1)\ldots w(2k)\end{pmatrix}(w(2k+1)\ldots w(2m)) \tag{A8}$$

where $\hat{S}_{2m-k,k}$ is the set of permutations of $\{k+1,\ldots,2m\}$ such that

$$w(k+1)<w(k+2)<\ldots<w(2k)\qquad\text{and}\qquad w(2k+1)<\ldots<w(2m).$$

Another definition of the Hall-Littlewood symmetric functions (Macdonald 1979) is

$$P_\lambda(\alpha_1,\ldots,\alpha_n;t)=\sum_{w\in S_n/S_n^\lambda}w\left(\alpha_1^{\lambda_1}\ldots\alpha_n^{\lambda_n}\prod_{\lambda_i>\lambda_j}\frac{\alpha_i-t\alpha_j}{\alpha_i-\alpha_j}\right) \tag{A9}$$

where $\lambda=(\lambda_1,\ldots,\lambda_m)$ with $m\leqslant n$ and $\lambda_{m+1}=\ldots=\lambda_n=0$. S_n^λ denotes the set of permutations of $\{1,\ldots,n\}$ which fix the monomial $\alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\ldots\alpha_n^{\lambda_n}$. It is known that $P_\lambda(\alpha_1,\ldots,\alpha_{n+1};t)=P_\lambda(\alpha_1,\ldots,\alpha_n,0;t)$ if $P(\lambda)\leqslant n$ and $P_{(\lambda_1,\ldots,\lambda_m,0)}=P_{(\lambda_1,\ldots,\lambda_m)}$ so, without loss of generality, we may assume that m and n are both even.

For $t=-1$, equation (2.7) implies that Q_λ, and hence by (2.10) P_λ also, vanish unless λ has distinct (non-zero) parts. Thus, for such λ

$$P_\lambda(\alpha_1,\ldots,\alpha_n;-1)$$
$$=\sum_{w\in\hat{S}_{n-m,m}}w\left[\left(\sum_{w'\in S_m}w'\left(\alpha_1^{\lambda_1}\ldots\alpha_m^{\lambda_m}\prod_{1\leqslant i<j\leqslant m}\frac{\alpha_i+\alpha_j}{\alpha_i-\alpha_j}\right)\right)\prod_{i=1}^{m}\prod_{j=m+1}^{n}\frac{\alpha_i+\alpha_j}{\alpha_i-\alpha_j}\right]$$

so that

$$\left(\prod_{1\leqslant i<j\leqslant n}\frac{\alpha_i-\alpha_j}{\alpha_i+\alpha_j}\right)P_\lambda(\alpha_1\ldots,\alpha_n;-1)$$
$$=\sum_{w\in\hat{S}_{n-m,m}}\varepsilon(w)w\left(\sum_{w'\in S_m}\varepsilon(w')w'(\alpha_1^{\lambda_1}\ldots\alpha_m^{\lambda_m})\prod_{m+1\leqslant i<j\leqslant n}\frac{\alpha_i-\alpha_j}{\alpha_i+\alpha_j}\right). \tag{A10}$$

For a $2p\times 2p$ matrix A with (i, j)th entry $(\alpha_i+\alpha_j)/(\alpha_i-\alpha_j)$, we may show that

$$\mathrm{Pf}(A)=\prod_{1\leqslant i<j\leqslant 2p}\left(\frac{\alpha_i-\alpha_j}{\alpha_i+\alpha_j}\right). \tag{A11}$$

This is because of the following properties are shared by the both sides of (A11). If any pair α_i, α_j $(1\leqslant i, j\leqslant 2p)$ are equal, then each side vanishes and from (A1) we see that the common denominator of Pf(A) is $\Pi_{i<j}(\alpha_i+\alpha_j)$. Also, the order of its numerator implies that Pf(A) has no zeros other than those described above and, by using induction, one can show that the coefficient on the RHS is unity.

Using (A11) in (A10), noticing that $\Sigma_{w'\in S_m}\varepsilon(w')w'(\alpha_1^{\lambda_1}\dots\alpha_m^{\lambda_m})$ is $\det(\alpha_i^{\lambda_j})$ and then by comparison with (A8), we find an expression for $P_\lambda(\alpha_1,\dots,\alpha_n;-1)$ as a ratio of Pfaffians. In this expression we perform a permutation of lines in order to eliminate the alternating factor and hence obtain

$$P_\lambda(\alpha_1,\dots,\alpha_n;-1)=\mathrm{Pf}(A')/\mathrm{Pf}(A) \tag{A12}$$

where $a_{i,j}=(\alpha_i+\alpha_j)/(\alpha_i-\alpha_j)$ so that A is a skew-symmetric $n\times n$ matrix and

$$A'=\begin{pmatrix} A & B\\ -B^{\mathrm{T}} & 0\end{pmatrix} \tag{A13}$$

where $b_{i,j}=(\alpha_i^{\lambda_j})$ and B is an $n\times m$ matrix. This result appears to be new and is an analogue of the famous expression for Schur functions as ratios of determinants. As an example of the Pfaffians in (A12), we take $\lambda=(21)$ and $n=3$. Since n is odd we extend the number of indeterminates to four so that, using the triangular array notation (see Caianiello 1973)

$$\mathrm{Pf}(A)=\left|\begin{array}{ccc} \dfrac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2} & \dfrac{\alpha_1-\alpha_3}{\alpha_1+\alpha_3} & \dfrac{\alpha_1-\alpha_4}{\alpha_1+\alpha_4}\\ & \dfrac{\alpha_2-\alpha_3}{\alpha_2+\alpha_3} & \dfrac{\alpha_2-\alpha_4}{\alpha_2+\alpha_4}\\ & & \dfrac{\alpha_3-\alpha_4}{\alpha_3+\alpha_4}\end{array}\right|$$

and so when $\alpha_4=0$,

$$\mathrm{Pf}(A)=\left|\begin{array}{ccc} \dfrac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2} & \dfrac{\alpha_1-\alpha_3}{\alpha_1+\alpha_3} & 1\\ & \dfrac{\alpha_2-\alpha_3}{\alpha_2+\alpha_3} & 1\\ & & 1\end{array}\right|$$

and

$$\mathrm{Pf}(A')=\left|\begin{array}{ccccc} \dfrac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2} & \dfrac{\alpha_1-\alpha_3}{\alpha_1+\alpha_3} & 1 & \alpha_1 & \alpha_1^2\\ & \dfrac{\alpha_2-\alpha_3}{\alpha_2+\alpha_3} & 1 & \alpha_2 & \alpha_2^2\\ & & 1 & \alpha_3 & \alpha_3^2\\ & & & 0 & 0\\ & & & & 0\end{array}\right|.$$

To prove (2.22) we consider (2.21) with $t=-1$ and use (A10) with indeterminates $\alpha_i = \partial^{(i)}$. This gives the formula

$$\Pi(\partial^{(i)}; -1)P_\mu(\partial^{(i)}; -1) = \mathrm{Pf}(A'(\partial^{(i)})) \tag{A14}$$

where $A'(\partial^{(i)})$ denotes the matrix A' with $\partial^{(i)}$ replacing α_i. In order to get this expression we replace the rational functions in (A10) with infinite power series, e.g.

$$\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2} = 1 - 2\alpha_1\alpha_2^{-1} + 2\alpha_1^2\alpha_2^{-2}\dots$$

and for $\alpha_i = \partial^{(i)}$ the RHS equals $(\partial^{(1)} - \partial^{(2)})(\partial^{(1)} + \partial^{(2)})^{-1}$ whenever this inverse is defined. In particular, as described following (2.8), (A14) is well defined when acting on $q_{\lambda_1}(\boldsymbol{x}^{(1)})\dots q_{\lambda_n}(\boldsymbol{x}^{(n)})$.

In (2.21) let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_m)$ be such that n and m are both even and, if not, append a zero to either partition so that they are so. Now, if we define $\lambda_{n+1} = \dots = \lambda_{n+m} = 0$, $\tilde{\mu}_j = 0$ for $j = 1 \dots n$ and $\tilde{\mu}_j = \mu_{n+m-j+1}$ for $j = n+1, \dots, n+m$, then

$$\mathrm{Pf}(A'(\partial^{(i)})) \prod_{i=1}^{n} q_{\lambda_i}(x^{(i)}; -1)|_{x^{(i)}=x} = 2^{|\mu|}\mathrm{Pf}(Q_{(\lambda_i-\tilde{\mu}_j,\lambda_j)}(\boldsymbol{x}; -1))$$

where (2.6) has been employed. Equation (2.22) now follows immediately.

In terms of the other notation for Pfaffians that we have used

$$P_\lambda(\alpha_1, \alpha_2, \dots \alpha_n; -1) = \frac{(1\ 2\ \dots\ n\ a_m\ \dots\ a_1)}{(1\ 2\ \dots\ n)} \tag{A15}$$

where

$$(ij) = \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \qquad (ia_j) = \alpha_i^{\lambda_j} \qquad (a_ia_j) = 0$$

while

$$Q_{\lambda/\mu}(\boldsymbol{x}; -1) = (1\ 2\ \dots n\ a_m \dots a_1) \tag{A16}$$

where

$$(ij) = Q_{(\lambda_i,\lambda_j)}(\boldsymbol{x}; -1) \qquad (ia_j) = q_{\lambda_i-\tilde{\mu}_j}(\boldsymbol{x}; -1) \qquad (a_ia_j) = 0.$$

This formula is essentially a generalisation of a result of Hirota (1989) expressed in different language. The proof here is quite different.

Finally in this appendix we list the matrices used in the transformation from derivatives of Q functions to skew Q functions. These are obtained using the transition matrices described by Macdonald (1979).

Let $X(t)$ be the square matrix with entries $X_\rho^\lambda(t)$, the constants in (2.15)-(2.17), in which the column index is λ and the row index is ρ. The dimension of $X(t)$ equals the number of partitions of $n = |\lambda| = |\rho|$. Such an $X(t)$ exists for each $n \geqslant 0$. In fact

$$X(t) = X(0)K(t)$$

where $K(t)$ are matrices of the same dimensions as $X(t)$ which are tabulated in Macdonald (1979) pp 126-7, and the entries in $X(0)$ are group characters χ_ρ^λ which may be found in the appendix of Littlewood (1950).

For $t=-1$ the Hall-Littlewood functions $P_\lambda(x;t)$ vanish unless λ is a partition into distinct parts, and are independent of x_{2k}. As a consequence of this, the matrices $X_\rho^\lambda(-1)$ are restricted to λ being a partition into distinct parts and ρ a partition into odd parts. The relevant restricted matrices $X(-1)$ are still square and for the given values of n are

$$n=1\quad \begin{array}{c} \\ (1) \end{array}\begin{array}{c} (1) \\ \begin{pmatrix} 1 \end{pmatrix} \end{array} \qquad n=2\quad \begin{array}{c} \\ (1^2) \end{array}\begin{array}{c} (2) \\ \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

$$n=3\quad \begin{array}{c} \\ (3) \\ (1^3) \end{array}\begin{array}{c} \begin{array}{cc} (3) & (21) \end{array} \\ \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} \end{array} \qquad n=4\quad \begin{array}{c} \\ (31) \\ (1^4) \end{array}\begin{array}{c} \begin{array}{cc} (4) & (31) \end{array} \\ \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \end{array}$$

$$n=5\quad \begin{array}{c} \\ (5) \\ (31^2) \\ (1^5) \end{array}\begin{array}{c} \begin{array}{ccc} (5) & (41) & (32) \end{array} \\ \begin{pmatrix} 1 & -2 & 2 \\ 1 & 0 & -1 \\ 1 & 3 & 2 \end{pmatrix} \end{array} \qquad n=6\quad \begin{array}{c} \\ (51) \\ (3^2) \\ (31^2) \\ (1^5) \end{array}\begin{array}{c} \begin{array}{cccc} (6) & (51) & (42) & (321) \end{array} \\ \begin{pmatrix} 1 & -1 & 0 & 2 \\ 1 & -2 & 2 & -4 \\ 1 & 1 & -1 & -1 \\ 1 & 4 & 5 & 2 \end{pmatrix} \end{array}.$$

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